Study of Maximally Entangled Mixed State: Case of Rank 2 and 3

Dissertation submitted to

Visvesvaraya National Institute of Technology, Nagpur

in partial fulfillment of the requirements for the award of the degree

Master of Science in Physics

by

Abhishek Dubey

under the guidance of

Dr. Udaysinh Bhosale

Department of Physics

Visvesvaraya National Institute of Technology, Nagpur

Nagpur 440 010, India

May 2025

Study of Maximally Entangled Mixed State: Case of Rank 2 and 3

Dissertation submitted to

Visvesvaraya National Institute of Technology, Nagpur

in partial fulfillment of the requirements for the award of the degree

Master of Science in Physics

by

Abhishek Dubey

under the guidance of

Dr. Udaysinh Bhosale

Department of Physics

Visvesvaraya National Institute of Technology, Nagpur

Nagpur 440 010, India

May 2025

© Visvesvaraya National Institute of Technology (VNIT), 2025

Department of Physics

Visvesvaraya National Institute of Technology, Nagpur

Dissertation Approval Sheet

Dissertation entitled "Study of Maximally Entangled Mixed State: Case of Rank 2 and 3" by Abhishek Dubey Enrollment No. **MS23PHY012** ID No. 30295 is approved for the degree of Master of Science in Physics

_____

Name:
Supervisor

_____

Name:
External Examiner

_____

Name:
Chairman

Department of Physics
Visvesvaraya National Institute of Technology, Nagpur

Declaration

I, **Abhishek Dubey**, hereby declare that this dissertation titled **"Study of Maximally Entangled Mixed States: Case of Rank 2 and 3"** is carried out by me in the Department of Physics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other Institution / University.

Abhishek Dubey
Date:

Certificate

This is to certify that the dissertation titled **"Study of Maximally Entangled Mixed States: Case of Rank 2 and 3"** is submitted by **Abhishek Dubey** in partial fulfillment of the requirements for the award of the degree of **Master of Science in Physics**, VNIT Nagpur. The work is comprehensive, complete and fit for final evaluation.

Head, Department of Physics
VNIT, Nagpur
Date:

Supervisor,
Dr. Udaysinh Bhosale
Assistant Professor
Dept. of Physics, VNIT, Nagpur

Visvesvaraya National Institute of Technology, Nagpur

Non- Plagiarism Certificate

I, **Abhishek Dubey**, hereby declare that this dissertation titled **"Study of Maximally Entangled mixed state: Case of Rank 2 and 3 "** is carried out by me in the Department of Physics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other Institution / University.

Abhishek Dubey
Date:

Dr. Udaysinh Bhosale
Assistant Professor
Department of Physics,
VNIT, Nagpur

turnitin
Page 2 of 24 - Integrity Overview
18% Overall Similarity
The combined total of all matches, including overlapping sources, for each database.

Acknowledgments

This project and thesis work would not have been possible without the kind support and help of many individuals and group of people. I would not have progressed in my project without their active direction, assistance, cooperation, and encouragement.

Firstly, I would like to thank my project guide **Dr. Udaysinh Bhosale , Assistant Professor, Department of Physics, VNIT, Nagpur (Maharashtra),** for his continuous support and motivation throughout the project work. It has been a great opportunity for me to work under his guidance on the academic front. I am also grateful for her constant advice, helping nature, and elderly support which made a significant difference in my life.

I am highly thankful to **Dr. B.R. Sankapal, Head of Department, Department of Physics, VNIT, Nagpur** and all my professors for their invaluable cooperation throughout the session.

I would also like to my friend Anup Das (M.Sc. student) for their motive support and sound guidance. They helped me in more ways than one in completing this work.

I would be remiss in not mentioning my parents Mr. Padmesh Dubey and Mrs. Sangeeta Dubey, and the whole family, who always encouraged me emotionally and financially. They always give me inspiration and hope for doing great in life.

Lastly, I would like to mention my friends and classmates for their cooperation. I am gratefully indebted for their valuable support and understanding.

Abstract

We know that in Quantum Information Theory, entanglement is a crucial resource and which states are maximally entangled by LOCC is essential. We have Bell states which are maximally entangled pure states but the situation is changes when we are dealing with the mixed states. Previous studies suggested that certain mixed states with the same spectrum could be treated as a maximally entangled states. But here in this thesis work we denied that. i.e. No MEMS for a fixed spectrum can exist always. Specifically, for a rank -2 and rank -3 mixed states , for a particular eigenvalue configuration no state can be transformed into all other isospectral states under LOCC or NE operations. This implies that - state maximizing a given entanglement measure among all states with the same spectrum depends on the choice of the entanglement measure, and thus, no single state can be maximally entangled across all measures. Moreover this result resolve the 25 years longstanding open problem by the contradictory method.

Contents

Contents **3**

1 Introduction **5**

1.1 Quantum Entanglement 5

1.2 Density Matrix 5

1.3 Condition for Separability and RDM 6

1.4 Bell Basis 8

1.5 D-dimensional Bell States 8

1.6 Local Operation Classical Communication (LOCC) 8

1.7 Fully entangled Fraction 8

2 The Case of Rank 3 **10**

2.1 Bell Diagonal States and Fully Entangled Fraction 10

2.2 Relative Entropy and Negativity 10

2.3 Non-Existence of NE Map 11

2.4 Contradiction via Separable State 12

3 The Case of Rank 3 **13**

3.1 Bell Diagonal States and Fully Entangled Fraction 13

3.2 Relative Entropy and Negativity 14

3.3 Non-Existence of NE Map 14

3.4 Contradiction via Separable State 15

4 The Case of Rank 2 **16**

4.1 Bell Diagonal States and Fully Entangled Fraction 16

4.2 Relative Entropy and Negativity 17

4.3 Non-Existence of NE Map 17

4.4 Contradiction via Separable State 18

5 Conclusion and Future Directions **19**

5.1 Conclusion 19

5.2 Future Directions 19

Bibliography **20**

Chapter 1

Introduction

1.1 Quantum Entanglement

It is a quantum correlation that exists between different subsystem of a composite quantum system. Now pure entanglement states are very fragile in nature. We find generally noisy entangled states through interaction With the environment. To achieve pure entanglement we need to manipulate entanglement by any sort of process either by global operation or local operations. However global operations are really difficult to implement if the distance between the parties are long enough. So we apply local operations -LOCC. Now, a density operator is a Hermitian linear operator ρ acting on (H) that satisfy two relation,

1. ρ is defined non negative.
2. Tr $(\rho) = 1$.

For pure states- $\rho^2 = \rho$ and For mixed states- $\rho^2 < \rho$,

Thus the pure states are identified with the vectors of H space where as Mixed states can be represented by probabilistic mixture of pure states [1].

1.2 Density Matrix

The density matrix ρ is represented as:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{pmatrix} \tag{1.1}$$

The density matrix ρ in bra-ket notation is:

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \tag{1.2}$$

where p_i are the probabilities and $|\psi_i\rangle$ are the quantum states.
Let ρ be a density matrix. The following are its key properties:

1. **Hermiticity:**

$$\rho^\dagger = \rho, \tag{1.3}$$

All eigenvalues of ρ are real.

2. **Normalization (Trace One):**

$$\mathrm{Tr}(\rho) = 1, \tag{1.4}$$

This ensures the total probability is 1.

3. **Semi-Definiteness of Positivity:**

$$\langle\psi|\rho|\psi\rangle \geq 0 \quad \text{for all } |\psi\rangle, \tag{1.5}$$

Equivalently, all eigenvalues $\lambda_i \geq 0$.

4. **Unitary Evolution (Closed Systems):**

$$\rho(t) = U(t)\rho(0)U^\dagger(t), \quad U(t) = e^{-iHt/\hbar}. \tag{1.6}$$

This preserves trace, Hermiticity, and positivity.

5. **Purity:**

$$\mathrm{Tr}(\rho^2) \leq 1. \tag{1.7}$$

- $\mathrm{Tr}(\rho^2) = 1$ for pure states.
- $\mathrm{Tr}(\rho^2) < 1$ for mixed states.

Minimum purity: $\mathrm{Tr}(\rho^2) = \frac{1}{d}$ for the maximally mixed state in a d-dimensional Hilbert space.

1.3 Condition for Separability and RDM

For a 2-qubit system, a state $|\psi\rangle$ is separable if and only if it can be written in the form:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \,. \tag{1.8}$$

If no such decomposition exists, the state is entangled.

Consider a general two-qubit state:

$$|\psi\rangle = \sum_{i,j=0}^{1} c_{ij} \, |i\rangle_A \otimes |j\rangle_B \tag{1.9}$$

The density matrix is given by:

$$\rho = |\psi\rangle \langle\psi| = \left(\sum_{i,j=0}^{1} c_{ij} \, |i\rangle_A \, |j\rangle_B \right) \left(\sum_{k,l=0}^{1} c_{kl}^* \, \langle k|_A \, \langle l|_B \right). \tag{1.10}$$

To compute the reduced density matrix for subsystem A, we take the partial trace over subsystem B:

$$\rho_A = \mathrm{Tr}_B(\rho), \tag{1.11}$$

In detail, this is:

$$\rho_A = \sum_{j=0}^{1} \langle j|_B \, \rho \, |j\rangle_B \,, \tag{1.12}$$

As an example, consider the Bell state:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right), \tag{1.13}$$

The corresponding density matrix is:

$$\rho = |\psi\rangle \langle\psi| = \frac{1}{2} \left(|00\rangle + |11\rangle \right) \left(\langle 00| + \langle 11| \right), \tag{1.14}$$

Now, tracing out subsystem B, we have:

$$\rho_A = \mathrm{Tr}_B \left(\frac{1}{2} \left(|00\rangle \langle 00| + |00\rangle \langle 11| + |11\rangle \langle 00| + |11\rangle \langle 11| \right) \right), \tag{1.15}$$

This simplifies to: $\rho_A = \frac{1}{2} \left(|0\rangle \langle 0| + |1\rangle \langle 1| \right)$ which is the maximally mixed state for qubit

$$\rho_A = \frac{1}{2} \left(|0\rangle \langle 0| + |1\rangle \langle 1| \right), \tag{1.16}$$

which is the maximally mixed state for qubit.

1.4 Bell Basis

The Bell basis states for a 2-qubit system in bra-ket notation are:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \tag{1.17}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \tag{1.18}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \tag{1.19}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \tag{1.20}$$

These four states form the Bell basis, which are Maximally Entangled States(MES) of two Q-bits [2].

1.5 D-dimensional Bell States

These are maximally entangled states of two qubits but can also be applicable on qudits(higher dimensions)

$$|\phi\rangle_d = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle. \tag{1.21}$$

1.6 Local Operation Classical Communication (LOCC)

Each side of experimentalist can perform quantum operations including Unitary transformation, State preparation and Measurement etc. on their own part of the system without any interaction with other side of the experimentalist but these parties are allowed to communicate with each other but only classically they cannot communicate quantum mechanically (Sending Q-bits).

1.7 Fully entangled Fraction

It is the defined as the maximal overlap with the maximally entangled pure states.

$$F(\rho) = \max_{U_A,U_B} \mathrm{Tr}(\rho, U_A \otimes U_B, \Phi_1, U_A^\dagger \otimes U_B^\dagger), \tag{1.22}$$

where the maximization is carried over arbitrary 2×2 unitaries U_A and U_B. If $F(\rho) > \frac{1}{2}$, then ρ must be known to be Entangled [3].

Basically it measures how a given state is close to MES.

Chapter 2

The Case of Rank 3

2.1 Bell Diagonal States and Fully Entangled Fraction

A state is called Bell diagonal if it is diagonal in the Bell basis, i.e.,

$$\rho = \sum_{i=1}^{4} p_i \Phi_i, \tag{2.1}$$

where Φ_i are the Bell states. These states are known to be entangled iff [4] ;

$$\max_i p_i > \frac{1}{2}. \tag{2.2}$$

We define the fully entangled fraction as:

$$F(\rho) = \max_{U_A, U_B} \mathrm{Tr}(\rho, U_A \otimes U_B, \Phi_1, U_A^\dagger \otimes U_B^\dagger), \tag{2.3}$$

where the maximization is over arbitrary 2×2 unitaries U_A, U_B. If $F(\rho) > \frac{1}{2}$, then ρ must be entangled [3].

So, for a separable state ρ, it must hold that

$$\mathrm{Tr}(\rho \Phi_i) \leq \frac{1}{2} \quad \text{for all } i \in 1, 2, 3, 4. \tag{2.4}$$

2.2 Relative Entropy and Negativity

We consider a class of states of the form:

$$\rho_{\vec{\lambda}} = \lambda_1 \Phi_1 + \lambda_2 \left|01\right\rangle \left\langle 01\right| + \lambda_3 \Phi_2 + \lambda_4 \left|10\right\rangle \left\langle 10\right|. \tag{2.5}$$

[5].

We now focus on rank-3 instances:

$$\rho_\lambda = \lambda_1\Phi_1 + \lambda_2\Phi_2 + (1-\lambda_1-\lambda_2)\left|10\right\rangle\left\langle 10\right| \tag{2.6}$$

$$\sigma_\lambda = \lambda_1\Phi_1 + \lambda_2\Phi_3 + (1-\lambda_1-\lambda_2)\Phi_4, \tag{2.7}$$

where $\lambda \in [\frac{1}{2}, 1]$. Note that ρ_λ and σ_λ will be isospectral for any choice of λ [6].

2.3 Non-Existence of NE Map

There is no NE (non-entangling) map Λ such that:

$$\Lambda(\rho_\lambda) = \sigma_\lambda \quad \text{for } \lambda \in (\tfrac{2}{3}, 1). \tag{2.8}$$

Assume such a map Λ exists. By linearity:

$$\Lambda(\rho_\lambda) = \lambda_1\Lambda(\Phi_1) + \lambda_2\Lambda(\Phi_2) + (1-\lambda_1-\lambda_2)\Lambda(\left|10\right\rangle\left\langle 10\right|) = \sigma_\lambda. \tag{2.9}$$

Taking trace with Φ_1 we get:

$$\lambda_1 = \mathrm{Tr}[\Phi_1\Lambda(\rho_\lambda)]. \tag{2.10}$$

$$= \lambda_1\mathrm{Tr}[\Phi_1\Lambda(\Phi_1)] + \lambda_2\mathrm{Tr}[\Phi_1\Lambda(\Phi_2)] + (1-\lambda_1-\lambda_2)\mathrm{Tr}[\Phi_1\Lambda(\left|10\right\rangle\left\langle 10\right|)] \quad \leq \lambda_1 + \lambda_2\mathrm{Tr}[\Phi_1\Lambda(\Phi_2)],$$

since the last term is ≤ 0.

Therefore,

$$\frac{1}{2} \leq \mathrm{Tr}[\Phi_1\Lambda(\Phi_2)]. \tag{2.11}$$

Similarly, using $\mathrm{Tr}[\Phi_3\sigma_\lambda]$:

$$\frac{1-\lambda_1-\lambda_2}{2\lambda_2} \leq \mathrm{Tr}[\Phi_3\Lambda(\Phi_2)]. \tag{2.12}$$

Also, from $\mathrm{Tr}[\Phi_4\sigma_\lambda]$:

$$\frac{-1+\lambda_1+3\lambda_2}{2\lambda_2} \leq \mathrm{Tr}[\Phi_4\Lambda(\Phi_2)]. \tag{2.13}$$

Adding these three:

$$\frac{2\lambda_2 - \lambda_1}{2\lambda_2} \leq \mathrm{Tr}[(\Phi_1 + \Phi_3 + \Phi_4)\Lambda(\Phi_2)]. \tag{2.14}$$

2.4 Contradiction via Separable State

Now consider the separable state:

$$\tau = \frac{1}{2}\Phi_2 + \frac{1}{4}\Phi_3 + \frac{1}{4}\Phi_4 = \frac{1}{2}\Phi_1 + \frac{1}{4}\left|01\right\rangle\left\langle 01\right| + \frac{1}{4}\left|10\right\rangle\left\langle 10\right|. \tag{2.15}$$

This is Bell-diagonal and hence separable. Now,

$$\mathrm{Tr}[\Phi_1\Lambda(\tau)] \geq \frac{1}{2} + \frac{1}{8} = \frac{5}{8}. \tag{2.16}$$

But for $\lambda > \frac{2}{3}$, this implies:

$$\mathrm{Tr}[\Phi_1\Lambda(\tau)] > \frac{1}{2}, \tag{2.17}$$

But if $\Lambda(\tau)$ is entangled (because its fully entangled fraction exceeds 1/2), this contradicts the assumption that Λ is NE (which must map separable states to separable states). [7] [8].

Chapter 3

The Case of Rank 3

3.1 Bell Diagonal States and Fully Entangled Fraction

A state is called Bell diagonal if it is diagonal in the Bell basis, i.e.,

$$\rho = \sum_{i=1}^{4} p_i \Phi_i. \tag{3.1}$$

Such a state is known to be entangled iff [4].

$$\max_i p_i > \frac{1}{2}. \tag{3.2}$$

Now, let us define the fully entangled fraction:

$$F(\rho) = \max_{U_A, U_B} \mathrm{Tr}\left(\rho \left(U_A \otimes U_B\right) \Phi_1 \left(U_A^\dagger \otimes U_B^\dagger\right)\right), \tag{3.3}$$

where the maximization is carried over arbitrary 2×2 unitaries U_A and U_B.

If $F(\rho) > \frac{1}{2}$, then ρ must be entangled [3].

So we have,

$$\mathrm{Tr}(\rho \Phi_i) \leq \frac{1}{2}, \quad \forall i \in \{1, 2, 3, 4\}, \tag{3.4}$$

must hold if ρ is separable.

3.2 Relative Entropy and Negativity

The relative entropy of entanglement and the negativity are found to be for the state

$$\rho_{\vec{\lambda}} = \lambda_1 \Phi_1 + \lambda_2 |01\rangle\langle 01| + \lambda_3 \Phi_2 + \lambda_4 |10\rangle\langle 10| . \tag{3.5}$$

We now consider the rank-3 instances of this state:

$$\begin{aligned} \rho_\lambda &= \lambda_1 \Phi_1 + \lambda_2 \Phi_2 + (1 - \lambda_1 - \lambda_2) |10\rangle\langle 10| , \\ \sigma_\lambda &= \lambda_1 \Phi_1 + \lambda_2 \Phi_3 + (1 - \lambda_1 - \lambda_2) \Phi_4. \end{aligned}$$

Here, $\lambda_1, \lambda_2 \in \left[\frac{1}{2}, 1\right]$, and ρ_λ and σ_λ are isospectral for any choice of λ.

3.3 Non-Existence of NE Map

We claim that there is no NE (non-entangling) map Λ such that

$$\Lambda(\rho_\lambda) = \sigma_\lambda, \quad \text{for } \lambda \in \left(\frac{2}{3}, 1\right). \tag{3.6}$$

Assume, for contradiction, that such a map Λ exists. By linearity of Λ:

$$\lambda_1 \Lambda(\Phi_1) + \lambda_2 \Lambda(\Phi_2) + (1 - \lambda_1 - \lambda_2) \Lambda(|10\rangle\langle 10|) = \sigma_\lambda. \tag{3.7}$$

Taking the trace with Φ_1 on both sides:

$$\lambda_1 = \lambda_1 \operatorname{Tr}[\Phi_1 \Lambda(\Phi_1)] + \lambda_2 \operatorname{Tr}[\Phi_1 \Lambda(\Phi_2)] + (1 - \lambda_1 - \lambda_2) \operatorname{Tr}[\Phi_1 \Lambda(|10\rangle\langle 10|)]. \tag{3.8}$$

Since Φ_1 is orthogonal to $|10\rangle\langle 10|$:

$$\Rightarrow \lambda_1 \leq \lambda_1 \operatorname{Tr}[\Phi_1 \Lambda(\Phi_1)] + \lambda_2 \operatorname{Tr}[\Phi_1 \Lambda(\Phi_2)]. \tag{3.9}$$

This implies:

$$\operatorname{Tr}[\Phi_1 \Lambda(\Phi_2)] \geq \frac{1}{2}. \tag{3.10}$$

Similarly, for $\operatorname{Tr}(\Phi_3 \sigma_\lambda)$:

$$\Rightarrow \frac{1 - \lambda_1 - \lambda_2}{2\lambda_2} \leq \operatorname{Tr}[\Phi_3 \Lambda(\Phi_2)]. \tag{3.11}$$

And for $\mathrm{Tr}(\Phi_4\sigma_\lambda)$:

$$\Rightarrow \frac{-1+\lambda_1+3\lambda_2}{2\lambda_2} \leq \mathrm{Tr}[\Phi_4\Lambda(\Phi_2)]. \tag{3.12}$$

Adding all three inequalities:

$$\Rightarrow \frac{2\lambda_2-\lambda_1}{2\lambda_2} \leq \mathrm{Tr}\left[(\Phi_1+\Phi_3+\Phi_4)\,\Lambda(\Phi_2)\right]. \tag{3.13}$$

3.4 Contradiction via Separable State

Now consider the state:

$$\tau = \frac{1}{2}\Phi_2 + \frac{1}{4}\Phi_3 + \frac{1}{4}\Phi_4 = \frac{1}{2}\Phi_1 + \frac{1}{4}\left|01\right\rangle\left\langle 01\right| + \frac{1}{4}\left|10\right\rangle\left\langle 10\right|. \tag{3.14}$$

This is a Bell diagonal state of the form $\rho = \sum_i p_i\Phi_i$ and is separable.

Suppose Λ is NE, then $\Lambda(\tau)$ must be separable. However, using the earlier inequalities:

$$\mathrm{Tr}[\Phi_1\Lambda(\tau)] \geq \frac{1}{2} + \frac{1}{8} = \frac{5}{8} > \frac{1}{2}. \tag{3.15}$$

But if $\Lambda(\tau)$ is entangled (because its fully entangled fraction exceeds 1/2), this contradicts the assumption that Λ is NE (which must map separable states to separable states).

Therefore, no such NE map Λ exists [7] [8].

Chapter 4

The Case of Rank 2

4.1 Bell Diagonal States and Fully Entangled Fraction

A state is called Bell diagonal if it is diagonal in the Bell basis, i.e.,

$$\rho = \sum_{i=1}^{4} p_i \Phi_i. \tag{4.1}$$

The above states are called entangled iff [4]

$$\max_i p_i > \frac{1}{2}. \tag{4.2}$$

.

Now let us define the fully entangled fraction:

$$F(\rho) = \max_{U_A, U_B} \mathrm{Tr}\left[\rho (U_A \otimes U_B) \Phi_1 (U_A^\dagger \otimes U_B^\dagger)\right], \tag{4.3}$$

where the maximization is carried over arbitrary 2×2 unitaries U_A and U_B.

If

$$F(\rho) > \frac{1}{2}, \tag{4.4}$$

then ρ must be Entangled [3].

Thus, in particular,

$$\mathrm{Tr}(\rho \Phi_i) \leq \frac{1}{2} \tag{4.5}$$

must hold for all $i \in \{1, 2, 3, 4\}$ if ρ is separable.

4.2 Relative Entropy and Negativity

The Relative Entropy of Entanglement and Negativity are studied using states of the form:

$$\rho_{\vec{\lambda}} = \lambda_1 \Phi_1 + \lambda_2 \left|01\right\rangle \left\langle 01\right| + \lambda_3 \Phi_2 + \lambda_4 \left|10\right\rangle \left\langle 10\right|. \tag{4.6}$$

We now consider the rank-2 instances of these states:

$$\rho_\lambda = \lambda \Phi_1 + (1-\lambda) \left|10\right\rangle \left\langle 10\right|, \quad \sigma_\lambda = \lambda \Phi_1 + (1-\lambda)\Phi_3, \tag{4.7}$$

where $\lambda \in \left[\frac{1}{2}, 1\right]$. Clearly, ρ_λ and σ_λ are isospectral for any λ.

4.3 Non-Existence of NE Map

We now show that there exists no NE map Λ such that:

$$\Lambda(\rho_\lambda) = \sigma_\lambda \quad \text{for } \lambda \in \left(\frac{2}{3}, 1\right). \tag{4.8}$$

Assume, for contradiction, that such an NE map Λ exists. By linearity:

$$\Lambda(\rho_\lambda) = \lambda \Lambda(\Phi_1) + (1-\lambda)\Lambda(\left|10\right\rangle \left\langle 10\right|) = \lambda \Phi_1 + (1-\lambda)\Phi_3. \tag{4.9}$$

Now taking the trace with Φ_1, we get:

$$\lambda = \lambda \operatorname{Tr}[\Phi_1 \Lambda(\Phi_1)] + (1-\lambda) \operatorname{Tr}[\Phi_1 \Lambda(\left|10\right\rangle \left\langle 10\right|)]. \tag{4.10}$$

Since Φ_1 is orthogonal to $\left|10\right\rangle \left\langle 10\right|$, the second term vanishes, and we obtain:

$$\operatorname{Tr}[\Phi_1 \Lambda(\Phi_1)] \geq 1. \tag{4.11}$$

Similarly, taking trace with Φ_3:

$$1 - \lambda = \lambda \operatorname{Tr}[\Phi_3 \Lambda(\Phi_1)] + (1-\lambda) \operatorname{Tr}[\Phi_3 \Lambda(\left|10\right\rangle \left\langle 10\right|)]. \tag{4.12}$$

Ignoring the second term again for a lower bound:

$$\frac{1-\lambda}{\lambda} \leq \operatorname{Tr}[\Phi_3 \Lambda(\Phi_1)]. \tag{4.13}$$

4.4 Contradiction via Separable State

Now consider the action of Λ on the separable Bell diagonal state:

$$\tau = \frac{1}{2}\Phi_1 + \frac{1}{4}\Phi_3 + \frac{1}{4}\Phi_4 = \frac{1}{2}\Phi_1 + \frac{1}{4}|01\rangle\langle 01| + \frac{1}{4}|10\rangle\langle 10|. \tag{4.14}$$

Since τ is a convex combination of Bell states with each weight $\leq \frac{1}{2}$, it is separable.

Now compute:

$$\mathrm{Tr}[\Phi_1\Lambda(\tau)] \geq \frac{1}{2} + \frac{1}{8} = \frac{5}{8}. \tag{4.15}$$

But if $\Lambda(\tau)$ is entangled (because its fully entangled fraction exceeds $1/2$), this contradicts the assumption that Λ is NE (which must map separable states to separable states).

Hence, such an NE map cannot exist for $\lambda > \frac{2}{3}$ [7] [8].

Chapter 5

Conclusion and Future Directions

5.1 Conclusion

We had shown that *maximally entangled mixed states* do not always exist within a fixed spectrum. Our analysis, particularly using the rank-2 and rank-3 approach with different choices of states, supports this result. Furthermore, we observe that the state which maximizes a given entanglement measure among all states with the same spectrum is not unique—it depends crucially on the choice of the entanglement measure itself.

5.2 Future Directions

- This work opens several avenues for further investigation. One key question is whether our findings hold under other notions of *the same level of noise*, especially those that arise in physically meaningful or operational contexts.
- Are there specific spectral distributions for which a maximally entangled state does exist, apart from the rank-1 (pure state) case?
- Can we identify other Mathematically or physically motivated definitions of the set S such that all states in S are guaranteed to be separable?
- How do different entanglement measures influence the geometry of the state space for fixed spectra?
- What implications do these findings have for entanglement distillation or quantum resource theories?

Answering these questions will leads us towards a more deeper understanding of the structure of mixed-state entanglement and its constraints under spectral limitations [2].

Bibliography

[1] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. Quantum entanglement. *Rev. Mod. Phys.*, 81:865–942, Jun 2009.

[2] Julio I. de Vicente. Maximally entangled mixed states for a fixed spectrum do not always exist. *Phys. Rev. Lett.*, 133:050202, Jul 2024.

[3] Michał Horodecki and Paweł Horodecki. Reduction criterion of separability and limits for a class of distillation protocols. *Phys. Rev. A*, 59:4206–4216, Jun 1999.

[4] Ryszard Horodecki and Michal/ Horodecki. Information-theoretic aspects of inseparability of mixed states. *Phys. Rev. A*, 54:1838–1843, Sep 1996.

[5] Frank Verstraete, Koenraad Audenaert, and Bart De Moor. Maximally entangled mixed states of two qubits. *Phys. Rev. A*, 64:012316, Jun 2001.

[6] M. A. Nielsen. Conditions for a class of entanglement transformations. *Phys. Rev. Lett.*, 83:436–439, Jul 1999.

[7] Fernando G. S. L. Brandao and Nilanjana Datta. One-shot rates for entanglement manipulation under non-entangling maps. *IEEE Transactions on Information Theory*, 57(3):1754–1760, 2011.

[8] Mario Berta, Fernando G. S. L. Brandão, Gilad Gour, Ludovico Lami, Martin B. Plenio, Bartosz Regula, and Marco Tomamichel. On a gap in the proof of the generalised quantum Stein's lemma and its consequences for the reversibility of quantum resources. *Quantum*, 7:1103, September 2023.